Properties of Sobolev–type metrics in the space of curves

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Abstract
We define a manifold $M$ where objects $c \in M$ are curves, which we parameterize as $c : S^1 \rightarrow \mathbb{R}^n$ ($n \geq 2$, $S^1$ is the circle).

We study geometries on the manifold of curves, provided by Sobolev–type Riemannian metrics $H^j$. These metrics have been shown to regularize gradient flows used in Computer Vision applications, see [13, 16] and references therein.

We provide some basic results of $H^j$ metrics; and, for the cases $j = 1, 2$, we characterize the completion of the space of smooth curves. We call these completions "$H^1$ and $H^2$ Sobolev–type Riemannian Manifolds of Curves". This result is fundamental since it is a first step in proving the existence of geodesics with respect to these metrics.

As a byproduct, we prove that the Fréchet distance of curves (see [7]) coincides with the distance induced by the “Finsler $L^\infty$ metric” defined in §2.2 in [18].

1 Introduction
Suppose that $c$ is an immersed curve $c : S^1 \rightarrow \mathbb{R}^n$, where $S^1 \subset \mathbb{R}^2$ is the circle; we want to define a geometry on $M$, the space of all such immersions $c$. The tangent space $T_c M$ of $M$ at $c$ contains all the deformations $h \in T_c M$ of the curve $c$, which are all the vector fields along $c$. Then, an infinitesimal deformation of the curve $c$ in “direction” $h$ will yield (to first order) the curve $c(u) + \varepsilon h(u)$. For the sake of simplicity, we postpone details of the definitions (in particular on the regularity of $c$ and $h$ and on the topology on $M$) to Section 2.

We would like to define a Riemannian metric on the manifold $M$ of immersed curves: this means that, given two deformations $h, k \in T_c M$, we want to define a scalar product $\langle h, k \rangle_c$, possibly dependent on $c$. The Riemannian metric would then entail a distance $d(c_0, c_1)$ between the curves in $M$, defined as the infimum of the length $\text{Len}(\gamma)$ of all smooth paths $\gamma : [0, 1] \rightarrow M$ connecting $c_0$ to $c_1$. We call a minimal geodesic a path providing the minimum of $\text{Len}(\gamma)$ in the class of $\gamma$ with fixed endpoints. (1)

At the same time, we would like to consider curves as “geometric objects;” i.e., curves up to reparameterization; to this end, we will define the space of geometrical curves as the quotient space $B \equiv M/\text{Diff}(S^1)$, that is the space of immersed curves up to reparameterization. For this reason we will ask that the metric defined on $M$ be

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(1) Note that this is an oversimplification of what we will actually do: compare definitions 10 and 12.
independent of the parameterization of the curves. $B$ and $M$ are the Shape Spaces that are studied in this paper.

§1.i Shape Theory

There are two different (but interconnected) fields of applications for Shape Theory in Computer Vision.

**Shape Optimization:** We want to find a shape that best satisfies a design goal. This is usually done by minimizing a chosen energy that is defined on shapes.

**Shape Analysis:** We study a family of shapes for purposes of computing statistics, (automatic) cataloging, probabilistic modeling, among others, and possibly to create an a-priori model for better Shape Optimization.

This entails an important remark.

**Remark 1** The aforementioned space $B$ represents “curves up to reparameterization.” A different approach would be to define the shape space $S$ as “curves up to reparameterization, rotation, translation, scaling...” which is often more convenient for Shape Analysis tasks. The Riemannian metrics we study are not defined on curves up to rotation, translation, scaling, etc., but they are invariant w.r.t. joint application of these actions in the sense that $d(g \circ c_0, g \circ c_1) = d(c_0, c_1)$ where $g$ is the action of rotation, translation, etc. Therefore the metric can be projected from $B$ to $S$.

If one wishes to have a consistent view of the geometry of the space of curves in both Shape Optimization and Shape Analysis, then one should use the same metric when computing distances, averages and morphs between shapes, as when optimizing with respect to shape. Consistency is especially important when optimizing an energy that contains an a-priori model obtained from a Shape Analysis study. In this case the optimization scheme has natural connections to the geometry of the shape space (see Section IIA in [14] for more details).

§1.ii Notation

We begin by introducing some notation. For $c : S^1 \to \mathbb{R}^n$ a smooth curve, let

$$\text{len}(c) \triangleq \int_{S^1} |\dot{c}(\theta)| \, d\theta$$

be the length of the curve $c$; we will often write $L = \text{len}(c)$, to shorten formulas.

For $g : S^1 \to \mathbb{R}^k$, we define the integration by arc-parameter

$$\int_{S^1} g(s) \, ds \triangleq \int_{S^1} g(\theta)|\dot{c}(\theta)| \, d\theta.$$ 

Let $D_s$ be the differential operator $D_s \triangleq \frac{1}{|\dot{c}(s)|} \partial_s$, (the derivative with respect to arclength), so that $D_s c$ is the tangent unit vector, and $D_s^2 c$ is the curvature of $c$, intended as a vector.
§1.iii Origin of the problem

A number of methods have been proposed in Shape Analysis to define distances between shapes, averages of shapes, and optimal morphings between shapes. At the same time, there has been much previous work in Shape Optimization (for example image segmentation via active contours and 3D stereo reconstruction via deformable surfaces). In these latter methods, many authors have defined energy functionals $E(c)$ on curves (or on surfaces), whose minima represent the desired segmentation/reconstruction, and have then utilized the calculus of variations to derive curve evolutions to flow toward local minimizers of $E(c)$, often referring to these evolutions as gradient flows. The reference to these flows as gradient flows implies a certain Riemannian metric on the space of curves, but this fact has been largely overlooked. We call this metric $H^0$, and define it by

$$\langle h, k \rangle_{H^0} = \frac{1}{L} \int_{S^1} \langle h(s), k(s) \rangle \, ds$$

where $h, k \in T_c M$, $L$ is the length of $c$, $ds = |\dot{c}(\theta)| \, d\theta$ is integration by arc-parameter, and $\langle h(s), k(s) \rangle$ is the usual Euclidean scalar product in $\mathbb{R}^n$ (which sometimes we also write as $h(s) \cdot k(s)$).

Unfortunately, gradient flows that are induced by the $H^0$ metric have many unpleasant properties and limitations.

**Example 2** Consider a family $C = C(\theta, t)$ evolving by the geometric heat flow (also known as motion by mean curvature)

$$\frac{\partial C}{\partial t} = D^2_s C.$$  

This well known flow is often referred to as the gradient flow for length; indeed, by direct computation we find that the $H^0$ gradient is

$$\nabla_{H^0} \text{len}(c) = \text{len}(c) D^2_s c$$

so the previous statement is true up to a conformal factor $1/\text{len}(c)$, that is,

$$\frac{\partial C}{\partial t} = -\frac{1}{\text{len}(C)} \nabla_{H^0} \text{len}(C)$$

It is important to remark that the geometric heat flow is well posed only for increasing time. This limits the usefulness of $H^0$ gradient flows in Shape Optimization, as illustrated in the following example.

**Example 3** Let $T = D_s c$ be the tangent vector of a planar curve $c$, and $L = \text{len}(c)$. We define the normal vector $N$ as the unit-length vector obtained by rotating $T$ counterclockwise by the angle $\pi/2$. We define the scalar curvature $\kappa$ so that $D^2_s c = \kappa N$. Let

$$\overline{c} \overset{def}{=} \frac{1}{L} \int_{S^1} c(s) \, ds$$

be the center of mass of the curve.

Let us fix a target point $v \in \mathbb{R}^2$. Let $E(c) \overset{def}{=} \frac{1}{2} |c - v|^2$ be a functional that penalizes the distance from the center of mass to $v$. By direct computation the $H^0$ gradient descent flow is

$$\frac{\partial c}{\partial t} = -\nabla_{H^0} E(c) = (\langle v - \overline{c}, N \rangle N - \kappa N \langle c - \overline{c}, (c - \overline{c}) \rangle).$$  \hfill (2)
Let $P \equiv \{ w : \langle (w - \tau) \cdot (v - \tau) \rangle \geq 0 \}$ be the half plane that is the region of the plane that is “on the $v$ side” w.r.t. $\tau$. This gradient descent flow (2) does move the center of mass towards the point $v$: indeed there is a first term $\langle (v - \tau) \cdot \nu \rangle \nu$ that moves the whole curve towards $v$; and a second term that tries to decrease the curve length out of $P$ and increase the curve length in $P$: and this is ill posed.

This is just one example of a large class of energies that may be of interest in Shape Optimization but whose $H^0$ gradient flow is ill-defined. A classical method to overcome such situations is to add a regularization term to the energy; this remedy, though, does change the energy, and ends up solving a different problem; see [13, 15, 16].

The situation is even worse when we consider Shape Analysis. Surprisingly, $H^0$ does not yield a well defined metric structure, since the associated distance is identically zero; this striking fact was first described in [9], and is generalized to spaces of submanifolds in [5]. So $H^0$ completely fails in our stated goal, which is to provide a geometry of the space of curves usable both for Shape Optimization and Shape Analysis.

§1.iii.1 Previous work

When the above problems were recognized, there were many attempts at finding a better metric for curves.

In [17, 18, 19] we proposed a set of desirable metric properties and discussed some models available in the literature. Eventually we proposed and studied conformal metrics such as

$$\langle h, k \rangle_{H^0} \overset{\text{def}}{=} \text{len}(c) \int \langle h(s), k(s) \rangle ds$$

and proved results regarding this metric. In particular, we showed that the associated distance is nondegenerate. We also proved that minimal geodesics exist if we restrict ourselves to only unit length curves with an upper bound on curvature.

The same approach was proposed independently by J. Shah in [10], who moreover proved that in the simplest case given by (3), minimal geodesics are represented by a curve evolution with constant speed along the normal direction.

Another possible definition appeared in [7] (by Michor and Mumford), who proposed the metric

$$\langle h, k \rangle_{H^0_A} \overset{\text{def}}{=} \int (1 + A\kappa^2(s)) \langle h(s), k(s) \rangle ds$$

where $\kappa$ is the curvature of $c$, and $A > 0$ is a fixed constant. They proved many results regarding this metric, in particular, that the induced distance is nondegenerate, and that the completion of smooth curves is in between the space $\text{Lip}$ of rectifiable curves, and the space $\text{BV}^2$ of rectifiable curves whose curvature is a bounded measure.

§1.iv Sobolev–type Riemannian Metrics

In [12] we proposed a family of Sobolev–type Riemannian Metrics.
Definition 4 Let \( c \in M \), \( L \) be the length of \( c \), and \( h, k \in T_c M \). Let \( \lambda > 0 \). We assume \( h \) and \( k \) are parameterized by the arclength parameter of \( c \). We define, for \( \lambda > 0 \) and \( j \geq 1 \) integer,
\[
\langle h, k \rangle_{H_j} \overset{\text{def}}{=} \langle h, k \rangle_{H^0} + \lambda L^2 \langle D_j^s h, D_j^s k \rangle_{H^0},
\]
\[
\langle h, k \rangle_{\tilde{H}_j} \overset{\text{def}}{=} \bar{h} \cdot \bar{k} + \lambda L^2 \langle D_j^s h, D_j^s k \rangle_{H^0}
\]
where again \( \bar{h} \overset{\text{def}}{=} \frac{1}{L} \int_S h(s) \, ds \) and \( D_j^s \) is the \( j \)-th derivative with respect to arclength.

Note that \( \langle h, k \rangle_{H^0} = h \cdot k \) so the difference in the two metrics above is in substituting the term \( h \cdot k \) by \( h \cdot k \).

It is easy to verify that the above definitions are inner products. Note that we have introduced length dependent scale factors so that these inner products (and corresponding norms) are independent of curve rescaling.

In the above paper, and in following papers [11, 14, 13, 15, 16] we studied how the use of Sobolev metrics positively impacts Shape Optimization tasks. Indeed we remark that changing the metric will change the gradient and thus the gradient descent flow. This change will alter the topology in the space of curves, but the change of topology does not affect the energy to be minimized nor its global minima.

In these papers we showed that the Sobolev–type gradients regularize the gradient flows of energies. This is exhibited in numerical experiments and applications, where it is observed that Sobolev flows will not in general be trapped in local minima due to small scale details or noise. It is also a mathematical property that a Sobolev metric will yield a lower degree gradient descent P.D.E than \( H^0 \) and will thereby be well-posed in many cases where the \( H^0 \) flow is ill-posed. We provide two simple examples

Example 5 If \( E(c) = |\tau - v|^2 \) is defined as in Example 3, the \( \tilde{H}^1 \) gradient \( f = \nabla_{\tilde{H}^1} E(c) \) of \( E \) is the unique function \( f \) satisfying
\[
\bar{f} = \bar{v} - v, \quad \lambda D_s f = D_s c \langle (\tau - v), (c - v) \rangle.
\]
The gradient descent flow is
\[
\frac{\partial C}{\partial t} = -\nabla_{\tilde{H}^1} E(C); \quad (5)
\]
the mean part
\[
\bar{\partial_t C} = v - C
\]
of this flow simply moves the whole curve so that the center of mass will move towards \( v \); the derivative part \( D_\partial C \) reparameterizes the curves according to the law
\[
\partial_\log(|\partial_\partial C|^2) = \langle D_\partial \partial_\partial C, D_\partial C \rangle = -\langle (C - v), (C - v) \rangle.
\]

Example 6 (§4.3 in [13]) In the case of the elastic energy \( E(c) = \int \kappa^2 \, ds = \int |D_s^2 c|^2 \, ds \), the \( H^0 \) gradient is \( \nabla_{H^0} E = LD_s(2D_3^L c + 3|D_s^2 c|^2D_s c) \), which includes fourth order derivatives; whereas the \( \tilde{H}^1 \)-gradient is
\[
-\frac{2}{\lambda L} c_{ss} - 3\lambda L(|D_s c|^2 D_s c) \ast \tilde{K}_\lambda
\]
so that the gradient descent flow is an integro-differential second order P.D.E. The kernel \( \tilde{K}_\lambda \) is defined in eqn. 17 in [13].
In conclusion, Sobolev gradient methods effectively enlarge the family of energies that may be used in Shape Optimization – without requiring extra regularization terms.

Since we did find great advantages by using Sobolev–type metrics in Shape Optimization, we would like to further analyze the properties of the related Riemannian Geometry. These metrics may indeed eventually satisfy the goal expressed in Section §1.i, that is, to provide a consistent geometry of the space of curves to be used both in Shape Optimization and in Shape Analysis.

One question of major interest is whether or not the Riemannian space of curves is complete? How can we characterize the completion of the space of smooth curves in the metric \( H^j \)? This question is a fundamental first step if we wish to prove that geodesics do exist, but it is also important in Shape Optimization since it would be a basic ingredient of any proof of existence and regularity for minimizing gradient flows.

§1.iv.1 Related works

A family of metrics similar to what we defined in Definition 4 (up to the length scale factors) was concurrently studied in [6]. In that paper the geodesic equation, horizontality, conserved momenta, lower and upper bounds on the induced distance and scalar curvatures are computed. In a much earlier work, Younes in [20] had proposed a computable definition of distance of curves, modeled on elastic curves. This model may be viewed as a Sobolev–type metric in the space of curves up to rotation, translation and scaling and has been studied in depth in a recent paper [8].

§1.v Paper outline

In the rest of this paper we present a mathematical study of the Riemannian geometry of curves defined in Definition 4, and specifically the cases \( j = 1, 2 \). In Section 2 we properly define the model space for the manifold of curves, and discuss benefits and shortcomings of different choices of hypotheses. In Section §2.i we define the Fréchet distance of curves (see [7]) and prove that it coincides with the distance induced by the “Finsler \( L^\infty \) metric” defined in Section 2.2 in [18]. In Section 3 we define \( H^j \) Sobolev–type Riemannian metrics and prove some basic properties. Eventually we characterize the completion of smooth curves in the \( H^1 \) and \( H^2 \) metric: those complete spaces are the “\( H^1 \) and \( H^2 \) Sobolev–type Riemannian Manifolds of Curves”.

2 Spaces of curves

As anticipated in the introduction, we want to define a geometry on \( M \), the space of all immersions \( c : S^1 \to \mathbb{R}^n \).

We will sometimes distinguish exactly what \( M \) is, choosing between the space \( \text{Imm}(S^1, \mathbb{R}^n) \) of immersions, the space \( \text{Imm}_f(S^1, \mathbb{R}^n) \) of free immersions, and \( \text{Emb}(S^1, \mathbb{R}^n) \) of embeddings. We recall that \( c : S^1 \to \mathbb{R}^1 \) is a free immersion when the only diffeomorphism \( \phi : S^1 \to S^1 \) satisfying \( c(u) = c(\phi(u)) \forall u \) is the identity. More details are in §2.4, §2.5 in [7].

We will equip \( M \) with a topology \( \tau \) stronger than the \( C^1 \) topology: then any such choice \( M \) is an open subset of the vector space \( C^1(S^1, \mathbb{R}^n) \) (that is a Banach space), so it is a manifold.

The tangent space \( T_c M \) of \( M \) at \( c \) contains vector fields \( h : S^1 \to \mathbb{R}^n \) along \( c \).
Note that we represent both curves $c \in M$ and deformations $h \in T_c M$ as functions $S^1 \to \mathbb{R}^n$; this is a special structure that is not usually present in abstract manifolds: so we can easily define “charts” for $M$:

**Remark 7 (Charts in $M$)** Given a curve $c$, there is a neighborhood $U_c$ of $0 \in T_c M$ such that for $h \in U_c$, the curve $c + h$ is still immersed; then this map $h \mapsto c + h$ is the simplest natural candidate to be a chart of $\Phi_c : U_c \to M$; indeed, if we pick another curve $\tilde{c} \in M$ and the corresponding $U_{\tilde{c}}$ such that $U_{\tilde{c}} \cap U_c \neq \emptyset$, then the equality $\Phi_c(h) = c + h = \tilde{c} + h = \Phi_{\tilde{c}}(h)$ can be solved for $h$ to obtain $h = (\tilde{c} - c) + h$.

The above is trivial but is worth remarking for two reasons: it stresses that the topology $\tau$ must be strong enough to maintain immersions; and is a basis block to what we will do in the space $B_{i,f}$ defined below.

We look mainly for metrics in the space $M$ that are independent of the parameterization of the curves $c$: to this end, we define these spaces of geometrical curves

$$B_i = B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

and

$$B_{i,f} = B_{i,f}(S^1, \mathbb{R}^2) = \text{Imm}_f(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

that are the quotients of the spaces $\text{Imm}_f$ and $\text{Imm}_f(S^1, \mathbb{R}^2)$ by $\text{Diff}(S^1)$; alternatively we may quotient by $\text{Diff}^+(S^1)$ (the space of orientation preserving automorphisms of $S^1$), and obtain spaces of geometrical oriented curves.

**Remark 8 (on model spaces and properties)** We have two possible choices in mind for the topology $\tau$ to put on $M$: the Fréchet space of $C^\infty$ functions; or a Hilbert space such as standard Sobolev space $H^3(S^1 \to \mathbb{R}^n)$.

Suppose we define on $M$ a Riemannian metric: we would like $B_i$ to have a nice geometrical structure; we would like our Riemannian Geometry to satisfy some useful properties.

Unfortunately, this currently seems an antinomy.

If $M$ is modeled on a Hilbert space $H^3$, then most of the usual calculus carries on: for example, the exponential map would be locally a diffeomorphism; but the quotient space $M / \text{Diff}(S^1)$ is not a smooth bundle, (since the tangent to the orbit contains $\dot{c}$ and this is in $H^{-1}$ in general!).

If $M$ is modeled on the Fréchet space of $C^\infty$ functions, then the quotient space $M / \text{Diff}(S^1)$ is a smooth bundle; but some of the usual calculus fails: the Cauchy-Lipschitz theorem of existence of local solutions to O.D.E.s does not hold in general; and the exponential map is not locally surjective.

We will suppose in the following that $\tau$ is the Fréchet space of $C^\infty$ functions; then $B_{i,f}$ is a manifold, the base of a principal fiber bundle while $B_i$ is not (see in §2.4.3 in [7] for details).

To define charts on this manifold, we imitate what was done for $M$:

**Proposition 9 (Charts in $B_{i,f}$)** Let $\Pi$ be the projection from $\text{Imm}_f(S^1, \mathbb{R}^2)$ to the quotient $B_{i,f}$.

Let $[c] \in B_{i,f}$: we pick a curve $c$ such that $\Pi(c) = [c]$. We represent the tangent space $T_{[c]}B_{i,f}$ as the space of all $k : S^1 \to \mathbb{R}^n$ such that $k(s)$ is orthogonal to $c(s)$.

Again we can define a simple natural chart $\Phi_{[c]}$ by projecting the chart $\Phi_c$ (defined in Remark 7): the chart is

$$\Phi_{[c]}(k) \overset{\text{def}}{=} \Pi(c(\cdot) + k(\cdot))$$
that is, it moves $c(u)$ in direction $k(u)$; and it is easily seen that the chart does not depend on the choice of $c$ such that $\Pi(c) = [\bar{c}]$. These maps $\Phi$ are a chart of $B_{i,j}$; the proof may be found in [7], or in §4.4.7 and §4.6.6 in [2].

We now define a Finsler metric $F$ on $M$; this is a lower semi continuous function $F : TM \to \mathbb{R}^+$ such that $F(c, \cdot)$ is a norm on $T_c M$, for all $c$.

If $\gamma : [0, 1] \to M$ is a path connecting two curves $c_0, c_1$, then we may define a homotopy $C : S^1 \times [0, 1] \to \mathbb{R}^n$ associated to $\gamma$ by $C(\theta, v) = \gamma(v)(\theta)$, and vice versa.

**Definition 10 (standard distance)** Given a metric $F$ in $M$, we could consequently define the standard distance of two curves $c_0, c_1$ as the infimum of the length

$$\int_0^1 F(\gamma(t), \dot{\gamma}(t)) \, dt$$

in the class of all $\gamma$ connecting $c_0, c_1$.

This is not, though, the most interesting distance for applications: we are indeed interested in studying metrics and distances in the quotient space $B \overset{\text{def}}{=} M/\text{Diff}(S^1)$.

We add an hypothesis on $F$.

**Definition 11** The metric $F(c, h)$ is “curve-wise parameterization invariant”, that is, it does not depend on the parameterization of the curves $c$.

If this is satisfied, then $F$ may be projected to $B \overset{\text{def}}{=} M/\text{Diff}(S^1)$; we will say that $F$ is a geometrical metric.

Consider two geometrical curves $[c_0], [c_1] \in B$, and a path $\gamma : [0, 1] \to B$, connecting $[c_0], [c_1]$: then we may lift it to a homotopy $C : S^1 \times [0, 1] \to \mathbb{R}^n$; in this case, the homotopy will connect a reparameterization $c_0 \circ \phi_0$ to a reparameterization $c_1 \circ \phi_1$, with $\phi_0, \phi_1 \in \text{Diff}(S^1)$. Since $F$ does not depend on the parameterization, we can factor out $\phi_0$ from the definition of the projected length.

To summarize, we define the

**Definition 12 (geometric distance)** Given $c_0, c_1$, we define the class $A$ of homotopies $C$ connecting the curve $c_0$ to a reparameterization $c_1 \circ \phi$ of the curve $c_1$, that is, $C(u, 0) = c_0(u)$ and $C(u, 1) = c_1(\phi(u))$. We define the geometric distance $d_F$ of $[c_0], [c_1]$ in $B \overset{\text{def}}{=} M/\text{Diff}(S^1)$ as the infimum of the length

$$\text{Len}_F(C) \overset{\text{def}}{=} \int_0^1 F(C(\cdot, v), \partial_v C(\cdot, v)) \, dv$$

in the class of all such $C \in A$.

Any homotopy that achieves the minimum of $\text{Len}_F(C)$ is called a geodesic.

We call such distances $d_F(c_0, c_1)$, dropping the square brackets for simplicity.\(^{(4)}\)

We provide an interesting example of the above ideas in the following section.

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\(^{(1)}\)If we use $\text{Diff}^+(S^1)$ to define $B$ then $\phi$ must be orientation preserving as well.

\(^{(2)}\)Note the difference between $\text{Len}(C)$ and $\text{len}(c)$, that was defined in eqn. (1).

\(^{(3)}\)We are abusing notation: these $d_F$ are not, properly speaking, distances in the space $M$, since the distance between $c$ and a reparameterization $c \circ \phi$ is zero.
§2.i \( L^\infty \)-type Finsler metric and Fréchet distance

We digress from the main theme of the paper to prove a result that will be used in the following. For any fixed immersed curve \( c \) and \( \theta \in S^1 \), we define for convenience

\[
\pi_N: \mathbb{R}^n \rightarrow \mathbb{R}^n \to \text{ the projection on the space } N(\theta) \text{ orthogonal to the tangent vector } D_s c(\theta),
\]

\[
\pi_N(\theta) w = w - \langle w, D_s c(\theta) \rangle D_s c(\theta) \quad \forall w \in \mathbb{R}^n. \tag{7}
\]

Consider two immersed curves \( c_0 \) and \( c_1 \); the Fréchet distance \( d_f \) (as found in [7]) is defined by

**Definition 13 (Fréchet distance)**

\[
d_f(c_0, c_1) \triangleq \inf_{\phi} \sup_u |c_1(\phi(u)) - c_0(u)|
\]

where \( u \in S^1 \) and \( \phi \) is chosen in the class of diffeomorphisms of \( S^1 \).

This is a well defined distance in the space \( B_i \) (that is not, though, complete w.r.t. this distance: its completion is the space of Fréchet curves).

Another similar distance was defined in §2.2 in [18] by a different approach, using a Finsler metric:

**Definition 14 (Finsler \( L^\infty \) metric)** If we wish to define a norm \( F(c, \cdot) \) on \( T_c M \) that is modeled on the norm of the Banach space \( L^\infty(S^1 \rightarrow \mathbb{R}^n) \), we define

\[
F^\infty(c, h) \triangleq \|\pi_N h\|_{L^\infty} = \sup_{\theta} |\pi_N(\theta) h(\theta)|
\]

We define the distance \( d^\infty(c_0, c_1) \) as in 12.

Section §2.2.1 in [18] discusses the relationship between the distance \( d^\infty \) and the Hausdorff distance of compact sets; we discuss here the relationship between \( d_f \) and \( d^\infty \): indeed we prove that \( d_f = d^\infty \).

**Theorem 15** \( d_f = d^\infty \).

**Proof.** Fix \( c_0 \) and \( c_1 \), and define \( \mathcal{A} \) as in Definition 12.

We recall that \( d^\infty \) is also equal to the infimum of

\[
d^\infty(c_0, c_1) = \inf_{C \in \mathcal{A}} \int_0^1 \sup_{\theta} \left| \frac{\partial C}{\partial v}(\theta, v) \right| \, dv
\]

as well (the proof follows immediately from prop. 3.10 in [18])

Consider a homotopy \( C = C(u, v) \in \mathcal{A} \) connecting the curve \( c_0 \) to a reparameterization \( c_1 \circ \phi \) of the curve \( c_1 \):

\[
\sup_u |c_1(\phi(u)) - c_0(u)| = \sup_u |C(u, 1) - C(u, 0)| = \sup_u \left| \int_0^1 \frac{\partial C}{\partial v}(u, v) \, dv \right| \leq \int_0^1 \sup_u \left| \frac{\partial C}{\partial v}(u, v) \right| \, dv
\]

so that \( d_f \leq d^\infty \).
On the other side, let
\[ C^\phi(\theta, v) \overset{\text{def}}{=} (1 - v)c_0(\theta) + vc_1(\phi(\theta)) \]
be the linear interpolation: then
\[ \frac{\partial C^\phi}{\partial v}(u, v) = c_1(\phi(u)) - c_0(u) \]
(that does not depend on \( v \)) so that
\[ \sup_u \left| \int_0^1 \frac{\partial C^\phi}{\partial v}(u, v) \, dv \right| = \int \sup_u \left| \frac{\partial C^\phi}{\partial v}(u, v) \right| \, dv \]
and then, for that particular homotopy \( C^\phi \),
\[ \text{Len}_\infty(C^\phi) = \sup_u |c_1(\phi(u)) - c_0(u)| \]
we compute the infimum of all possible choices of \( \phi \) and get that
\[ d_\infty(c_0, c_1) = \inf_{C^\phi} \text{Len}_\infty(C^\phi) = \inf_{\phi} \sup_u |c_1(\phi(u)) - c_0(u)| = d_f(c_0, c_1) \]
\[ \square \]
The theorem holds as well if we use orientation preserving diffeomorphism \( \text{Diff}^+(S^1) \) both in the definition of the Fréchet distance and in the definition of \( L^\infty \).

### 3 Sobolev-type \( H^j \) metrics

We start by generalizing the definition 4. Fix \( \lambda > 0 \). Suppose that \( h \in L^2 \), then we can express it in Fourier series:
\[ h(s) = \sum_{l \in \mathbb{Z}} \hat{h}(l) \exp \left( \frac{2\pi i}{L}s \right) \]
where \( \hat{h} \in \ell^2(\mathbb{Z} \to \mathbb{C}) \).

For any \( \alpha > 0 \), given the Fourier coefficients \( \hat{h}, \hat{k} : \mathbb{Z} \to \mathbb{C} \) of \( h, k \), we define the fractional Sobolev inner product
\[ \langle h, k \rangle_{H^\alpha_0} \overset{\text{def}}{=} \sum_{l \in \mathbb{Z}} (2\pi l)^{2\alpha} \hat{h}(l) \cdot \overline{\hat{k}(l)} \]
that is independent of curve scaling; then we can define
\[ \langle h, k \rangle_{H^\alpha} \overset{\text{def}}{=} \overline{h} \cdot k + \lambda \langle h, k \rangle_{H^\alpha_0} \]
\[ \langle h, k \rangle_{\tilde{H}^\alpha} \overset{\text{def}}{=} \overline{h} \cdot \overline{k} + \lambda \langle h, k \rangle_{H^\alpha_0} \]
(10)

When \( \alpha = j \) integer, these definition coincide with the one in 4. So, for any \( \alpha > 0 \), we represent the Sobolev--type metrics by
\[ \langle h, k \rangle_{H^\alpha} = \sum_{l \in \mathbb{Z}} (1 + \lambda(2\pi l)^{2\alpha}) \hat{h}(l) \cdot \overline{\hat{k}(l)} \]
(11)
\[ \langle h, k \rangle_{\tilde{H}^\alpha} = \hat{h}(0) \cdot \overline{\hat{k}(0)} + \sum_{l \in \mathbb{Z}} \lambda(2\pi l)^{2\alpha} \hat{h}(l) \cdot \overline{\hat{k}(l)}. \]
(12)
§3.i Preliminary results

Remark 16 Unfortunately for \( j \) that is not an integer, the inner products (therefore, norms) are not local, that is, they cannot be written as integrals of derivatives of the curves. An interesting representation is by kernel convolution: given \( r \in \mathbb{R}^+ \), we can represent them, for \( j \) integer \( j > r + 1/4 \), as

\[
\langle h, k \rangle_{\tilde{H}^r} = \int_{c} \int_{c} D^j h(s) K(s - \tilde{s}) D^j k(\tilde{s}) \, ds \, d\tilde{s}
\]

that is, \( \langle h, k \rangle_{\tilde{H}^r} = \langle D^j h, K * D^j k \rangle_{H^0} \), for a specific kernel \( K \). Here \( * \) denotes convolution in \( S^1 \) w.r.t. arc parameter.

Remark 17 The norm \( \|h\|_{\tilde{H}^r} \) has an interesting interpretation in connection with applications in Computer Vision.

Consider a deformation \( h \in T_c M \) and write it as \( h = \tilde{h} + \hat{h} \): this decomposes

\( T_c M = \mathbb{R}^n \oplus D_c M \) \hspace{1cm} (13)

with

\[ D_c M \overset{\text{def}}{=} \left\{ h : S^1 \to \mathbb{R}^n \mid \tilde{h} = 0 \right\} \]

If we assign to \( \mathbb{R}^n \) its usual Euclidean norm, and to \( D_c M \) the scale-invariant \( H_0^\alpha \) norm defined in eqn. (9), then we are naturally lead to decompose as in eqn. (10), that is

\[ \|h\|^2_{H^\alpha} = \|\tilde{h}\|^2_{H_0^\alpha} + \lambda \|\hat{h}\|^2_{\tilde{H}^\alpha} \] \hspace{1cm} (14)

This means that the two spaces \( \mathbb{R}^n \) and \( D_c M \) are orthogonal w.r.t. \( \tilde{H}^\alpha \).

In the above, \( \mathbb{R}^n \) is akin to be the space of translations and \( D_c M \) the space of non-translating deformations. That labeling is not rigorous, though! since the subspace of \( T_c M \) that does not move the center of mass \( c \) is not \( D_c M \), but rather

\[ \left\{ h : \int_{S^1} h + (c - \tilde{c}) (D_c h \cdot T) \, ds = 0 \right\} . \]

Note that \( \sqrt{\langle h, h \rangle_{H_0^\alpha}} \) is a norm on \( D_c M \) (by (16)), and it is a seminorm and not a norm on \( T_c M \).

We define Finsler norms as

\[ F_{H^\alpha}(c, h) = \|h\|_{H^\alpha} = \sqrt{\langle h, h \rangle_{H^\alpha}} , \quad F_{\tilde{H}^\alpha}(c, h) = \|h\|_{\tilde{H}^\alpha} = \sqrt{\langle h, h \rangle_{\tilde{H}^\alpha}} \]

and consequently we define distances \( d_{H^\alpha} \) and \( d_{\tilde{H}^\alpha} \), as explained in Definition 12.

§3.i Preliminary results

We improve a result from [12] \( ^5 \): we show that the norms associated with the inner products \( H^\alpha \) and \( \tilde{H}^\alpha \) are equivalent. We first prove

Lemma 18 (Poincaré inequalities) Pick \( h : [0, L] \to \mathbb{R}^n \), weakly differentiable, with \( h(0) = h(L) \) (so \( h \) is periodically extensible): then

\[
\sup_u |h(u) - \tilde{h}| \leq \frac{1}{2} \int_0^L |h'(s)| \, ds . \] \hspace{1cm} (15)

\( ^5 \) and we provide a better version that unfortunately was prepared too late for the printed version of [12]
where \( \mathbb{1}_A(x) \) is the characteristic function, taking value 1 for \( x \in A \), 0 for \( x \notin A \). The constant \( 1/2 \) is optimal and is approximated by a family of \( h \) such that
\[
h'(s) = a(\mathbb{1}_{[0,\varepsilon)}(s) - \mathbb{1}_{[\varepsilon,2\varepsilon)}(s))
\]
when \( \varepsilon \to 0 \) (for a fixed \( a \in \mathbb{R}^n \)).

**Proof.** Since \( h(0) = h(L) \)
\[
h(u) - h(0) = \int_0^u h'(s) \, ds = -\int_u^L h'(s) \, ds
\]
then derive these equations
\[
h(u) - h(0) = \frac{1}{2} \left( \int_0^u h'(s) \, ds - \int_u^L h'(s) \, ds \right) \Rightarrow
\]
\[
\Rightarrow \, \overline{h} - h(0) = \frac{1}{2L} \int_0^L \left( \int_0^u h'(s) \, ds - \int_u^L h'(s) \, ds \right) \, du \Rightarrow
\]
\[
\Rightarrow \, |\overline{h} - h(0)| \leq \frac{1}{2L} \int_0^L \left( \int_0^u |h'(s)| \, ds + \int_u^L |h'(s)| \, ds \right) \, du = \frac{1}{2L} \int_0^L \left( \int_0^L |h'(s)| \, ds \right) \, du = \frac{1}{2} \int_0^L |h'(s)| \, ds
\]
so that (by extending \( h \) and replacing 0 with an arbitrary point) we prove (15). \( \square \)

**Corollary 19** By using Hölder inequality we can then derive many useful Poincaré inequalities of the form \( \|h - \overline{h}\|_p \leq c_{p,q,j} \|h'\|_q \). By Fourier transform we can also prove for \( p = q = 2 \) that
\[
\int_0^L |h(s) - \overline{h}|^2 \, ds \leq \frac{L^{2j}}{(2\pi)^j} \int_0^L |h^{(j)}(s)|^2 \, ds
\]
where the constant \( c_{2,2,j} = (L/2\pi)^{2j} \) is optimal and is achieved by \( h(s) = a \sin(2\pi s/L) \) (with \( a \in \mathbb{R}^n \)).

**Proposition 20**
\[
\|h\|_{\overline{H}^j} \leq \|h\|_{H^j} \leq \sqrt{1 + \frac{(2\pi)^{2j}\lambda}{(2\pi)^{2j}\lambda}} \|h\|_{\overline{H}^j}.
\]

**Proof.** Fix a smooth immersed curve \( c : S^1 \to \mathbb{R}^n \). By Hölder’s inequality, we have that \( |\overline{h}|^2 \leq \frac{1}{L} \int_0^L |h(s)|^2 \, ds \) so that \( \|h\|_{\overline{H}^j} \leq \|h\|_{H^j} \). On the other hand,
\[
\frac{1}{L} \int_0^L |h(s) - \overline{h}|^2 \, ds = \frac{1}{L} \int_0^L |h(s)|^2 \, ds - |\overline{h}|^2
\]
so that (by the Poincaré inequality (16)),
\[
\|h\|_{H^j}^2 = \int_0^L \frac{1}{L} |h(s)|^2 + \lambda L^{2j-1} |h^{(j)}(s)|^2 \, ds
\]
\[
= \frac{1}{L} \int_0^L |h(s) - \overline{h}|^2 \, ds + \int_0^L \lambda L^{2j-1} |h^{(j)}(s)|^2 \, ds + |\overline{h}|^2
\]
\[
\leq |\overline{h}|^2 + L^{2j-1} \left( \frac{1}{(2\pi)^{2j}} + \lambda \right) \int_0^L |h^{(j)}(s)|^2 \, ds \leq \frac{1 + (2\pi)^{2j}\lambda}{(2\pi)^{2j}\lambda} \|h\|_{\overline{H}^j}^2.
\]
§3.i Preliminary results

More generally

**Proposition 21** For \(i = 0, \ldots, j\), choose \(\overline{a}_0 \geq 0\) and \(a_i \geq 0\) with \(a_0 + \overline{a}_0 > 0\) and \(a_j > 0\). Define a \(H^j\)-type Riemannian norm

\[
\|h\|_{(a),j}^2 := \overline{a}_0|\overline{h}|^2 + \sum_{i=0}^j a_i L^{2i-1} \int_0^L |h^{(i)}(s)|^2 ds \tag{18}
\]

then all such norms are equivalent.

Moreover, choose \(r\) with \(1 \leq r \leq j\), and choose \(\overline{b}_0 \geq 0\), \(b_i \geq 0\) with \(\overline{b}_0 + b_0 > 0\), \(b_r > 0\): then the norm \(\|h\|_{(a),j}\) is stronger than the norm \(\|h\|_{(b),r}\).

**Proof.** The proof is just an application of (17) and of (16) (repeatedly); note also that for \(1 \leq i < j\) equation (16) becomes

\[
\int_0^L |h^{(i)}(s)|^2 ds \leq \frac{L^{2j-2i}}{(2\pi)^{2j-2i}} \int_0^L |h^{(j)}(s)|^2 ds \tag{19}
\]

since \(h^{(j)} = 0\).

So our definitions of \(\| \cdot \|_{H^j}\) and \(\| \cdot \|_{\tilde{H}^j}\) are in a sense the simpler choices of a Sobolev type norm that are scale invariant; in particular,

**Remark 22** the \(H^j\) type metric

\[
\|h\|_M^2 := \int \sum_{i=0}^j |h^{(i)}(s)|^2 ds
\]

studied in [6] is equivalent to our choices,

\[
b_1 \| \cdot \|_{H^j} \leq \| \cdot \|_M \leq b_2 \| \cdot \|_{\tilde{H}^j}
\]

but the constants \(b_1, b_2\) depend on the length of the curve.

Following these propositions, we will prove some properties of the \(H^1\) metric, and we will know that they can be extended to \(\tilde{H}^1\) and to more general \(H^j\)-type metrics defined as in (18).

We prove this fundamental inequality (20):

**Proposition 23** Let \(C(u, v)\) be a smooth homotopy of immersed curves \(C(\cdot, v)\), then

\[
\|\partial_v C(\cdot, v)\|_{H^1} \geq \sqrt{\lambda} \int |\partial_u C(u, v)| du . \tag{20}
\]

\(^{(6)}\)the scalar product can be easily inferred, by using polarization.
**Proof.** Fix a smooth immersed curve $c : S^1 \rightarrow \mathbb{R}^n$, let $L = \text{len}(c)$. Let $h : S^1 \rightarrow \mathbb{R}^n$ be a vector field. We rewrite for convenience

$$
\|h\|_{H^1}^2 \geq \lambda L^2 \langle h', h' \rangle_{H^0} = \lambda L \int_0^L |h'(s)|^2 \, ds = \lambda \int |\dot{c}(u)| \, \int |h'(u)|^2 |\dot{c}(u)| \, du
$$

where $h' = D_s h$; then by Cauchy-Schwartz

$$
\int |\dot{c}(u)| \, du \int |h'(u)|^2 |\dot{c}(u)| \, du \geq \left( \int |h'(u)| |\dot{c}(u)| \, du \right)^2.
$$

To conclude, set $h(u, v) = \partial_v C(u, v)$ so that $D_s h = D_s \partial_v C = \partial_{uv} C/|\partial_u C|$. □

As argued in Proposition 21, the above result extends to all $H^j$-type norms (18).

We here relate the $H^1$-type metric to the $L^\infty$ type metrics.

**Proposition 24** The $\tilde{H}^1$ metric is stronger than the $L^\infty$ metric defined in Definition 14.

As a consequence, by Theorem 20 and 15, the $H^j$ and $\tilde{H}^j$ distances are lower bounded by the Fréchet distance (with appropriate constants depending on $\lambda$).

**Proof.** Indeed, by (15) there follows

$$
\sup_\theta |\pi_N(\theta) h(\theta)| \leq \sup_\theta |h(\theta)| \leq |\tilde{h}| + \frac{1}{2} \int |h'| \, ds \leq |\tilde{h}| + \sqrt{L} \sqrt{\int |h'|^2 \, ds} \leq \sqrt{2} |\tilde{h}| + \frac{L}{4} \int |h'|^2 \, ds
$$

($\pi_N$ was defined in eqn. (7)). For example, choosing $\lambda = 1/4$,

$$
F_{\infty}(c, h) \leq \sqrt{2} \|h\|_{\tilde{H}^1}.
$$

□

We also establish relationship between the length $\text{len}(c)$ of a curve and the Sobolev metrics.

**Proposition 25** Suppose again that $C(u, v)$ is a smooth homotopy of immersed curves, let $L(v) \overset{\text{def}}{=} \text{len}(C(\cdot, v))$ be the length at time $v$; then

$$
\partial_v L = \left\langle \partial_{uv} C, \frac{\partial_u C}{|\partial_u C|} \right\rangle \, du \leq \int |\partial_{uv} C| \, du \leq \frac{1}{\sqrt{\lambda}} \|C(\cdot, v)\|_{H^1},
$$

by (20).

We have many interesting consequences:

- 

$$
|L(1) - L(0)| \leq \frac{1}{\sqrt{\lambda}} \text{Len}(C)
$$

where the length $\text{Len}(C)$ of the homotopy/path $C$ is computed using either $H^1$ or $\tilde{H}^1$ (or using any metric as in (18) above, but in this case the constant in (21) would change).
• Define the length functional $c \mapsto \text{len}(c)$ on our space of curves; embed the space of curves with a $H^1$ metric; then the length functional is Lipschitz.

• The “zero curves” are the constant curves (that have zero length); these are points in the space of curves where the space of curves is, in a sense, singular; by the above, the “zero curves” are a closed set in the $H^1$ space of curves, and an immersed curve $c$ is distant at least $\text{len}(c)\sqrt{\lambda}$ from the “zero curves”.

But the most interesting consequence is that

Theorem 26 (Completion of $B_1$ w.r.t. $H^1$) let $d_{H^1}$ be the distance induced by $H^1$; the metric completion of the space of curves is contained in the space of all rectifiable curves.

Proof. This statement is a bit fuzzy: indeed $d_{H^1}$ is not a distance on $M$, whereas in $B$ objects are not functions, but classes of functions. So it must be intended “up to reparameterization of curves”, as follows. (7)

Let $(c_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Since $d_{H^1}$ does not depend on parameterization, we assume that all $c_n$ are parameterized by arc parameter, that is, $|\partial_{\theta}c_n| = l_n$ constant in $\theta$. By Proposition 24, all curves are contained in a bounded region; since $\text{len}(c_n) = 2\pi l_n$ by Proposition 25 above, the sequence $l_n$ is bounded. So the (reparameterize) family $(c_n)$ is equibounded and equilipschitz: by Ascoli-Arzelà theorem, up to a subsequence, we obtain that $c_n$ converges uniformly to a Lipschitz curve $c$, and $|\partial_{\theta}c| \leq \lim_n l_n$.

We also prove that

Theorem 27 Any rectifiable planar curve is approximable by smooth curves according to the distance induced by $H^1$.

To prove this theorem, we will need a preliminary lemma taken from [18].

Let in the following $c$ be a rectifiable curve, and assume that it is non-constant. We identify $S^1$ with $[0, 2\pi)$. Let also in the following $L^2 = L^2([0, 2\pi])$.

Lemma 28 Suppose that $|\partial_{\theta}c| = 1$. We define the measurable angle function to be a function $\tau : [0, 2\pi) \rightarrow [0, 2\pi)$ such that $\partial_{\theta}c(\theta) = (\cos \tau(\theta), \sin \tau(\theta))$. We define

$S = \{ \tau \in L^2([0, 2\pi]) \mid \phi(\tau) = (0, 0) \}$

where $\phi : L^2 \rightarrow \mathbb{R}^2$ is defined by

\[
\phi_1(\tau) = \int_0^{2\pi} \cos \tau(s)ds , \quad \phi_2(\tau) = \int_0^{2\pi} \sin \tau(s)ds
\]

(this is similar to what was done in Srivastava et al. works on “Shape Representation using Direction Functions”, see [3]).

i). Assume that $c$ is not flat, that is, the image of $c$ is not contained in a line in the plane. Then, by the implicit function theorem, $S$ is a smooth immersed submanifold of codimension 2 in $L^2$, locally near $\tau$. (7)
ii). Moreover there exists a smooth projection \( \pi : V \to S \) defined in a neighborhood \( V \subset L^2 \) of \( \tau \) such that, if \( f(s) \) is smooth in \( s \), then \( \pi(f)(s) \) is smooth in \( s \).

**Proof.** i). The proof is a simple adaptation of Proposition 2.12 in [18]. Suppose by contradiction that \( \nabla \phi_1, \nabla \phi_2 \) are linear dependant at \( \theta \subset M \), that is, there exists \( a \in \mathbb{R}^2, a \neq 0 \) s.t.

\[
 a_1 \cos(\tau(\theta)) + a_2 \sin(\tau(\theta)(s)) = 0
\]

this means that, at all \( \theta, \partial \phi e \) is orthogonal to \((a_1, a_2)\), that implies that \( c \) is a flat curve. So, if \( c \) is not flat, then by the Implicit Function Theorem (5.9 in [4]) \( S \) is a smooth immersed submanifold of \( L^2 \), locally near \( \tau \).

ii). We adapt part of the proof of Proposition 2.15 in [18]. Fix \( \tau_0 \in S \) associated to a non flat curve \( c_0 \). Let \( T = T_{\tau_0}S \) be the tangent at \( \tau_0 \). \( T \) is the vector space orthogonal to \( \nabla \phi_i(\tau_0) \) for \( i = 1, 2 \). Let \( e_i = e_i(s) \in L^2 \cap C^\infty \) be near \( \nabla \phi_i(\tau_0) \) in \( L^2 \), so that the map \((x, y) : T \times \mathbb{R}^2 \to L^2 \)

\[
 (x, y) \mapsto \tau = \tau_0 + x + \sum_{i=1}^2 e_i y_i
\]

is a linear isomorphism. Let \( M' \) be \( M \) in these coordinates; by the Implicit Function Theorem, there exists an open set \( U' \subset T \), \( 0 \in U' \), an open \( V' \subset \mathbb{R}^2 \), \( 0 \in V' \), and a smooth function \( f : U \to \mathbb{R}^2 \) such that the local part \( M' \cap (U' \times V') \) of the manifold \( M' \) is the graph of \( y = f(x) \).

We immediately define a smooth projection \( \pi' : U' \times V' \to M' \) by setting \( \pi'(x, y) = (x, f(x)) \). This may be expressed in \( L^2 \); let \((x(\tau), y(\tau))\) be the inverse of (22) and \( U = x^{-1}(U') \); we define the projection \( \pi : U \to M \) by setting

\[
 \pi(\tau) = \tau_0 + x + \sum_{i=1}^2 e_i f_i(x(\tau)) .
\]

Then

\[
 \pi(\tau)(s) - \tau(s) = \sum_{i=1}^2 e_i(s) a_i , \quad a_i \equiv (f_i(x(\tau)) - y_i) \in \mathbb{R}
\]

so if \( \tau(s) \) is smooth, then \( \pi(\tau)(s) \) is smooth.

\( \square \)

**Proof of 27.** We sketch how we can approximate \( c \) by smooth curves. Since the metric is independent of reparameterization and rescaling, we rescale \( c \) and assume that \( |\partial c| = 1 \).

As a first step, we assume that \( c \) is not flat; then, by Lemma 28, \( S \) is a manifold near \( \tau \); and let \( \pi \) be as in the above lemma.

Let \( f_n \) be a sequence of smooth functions with \( f_n \to \tau \) in \( L^2 \); then \( g_n \equiv \pi(f_n) \to \tau \). Let then

\[
 G_n(\theta, t) \equiv \pi(t \tau + (1 - t)g_n)(\theta)
\]

be a the projection on \( S \) of the linear path connecting \( \tau \) to \( g_n \). Since \( S \) is smooth in \( V \), then the \( L^2 \) distance \( ||\tau - g_n|| \) is equivalent to the geodesic induced distance; in particular,

\[
 \lim_n \mathbb{E}S(G_n) = 0
\]
§3.i Preliminary results

where
\[ E_S(G) \overset{\text{def}}{=} \int_0^1 \| \partial_t G(\cdot, t) \|^2_{L^2} \, dt \]
is the action of the path \( G \) in \( S \subset L^2 \).

The above \( G_n \) can be associated to an homotopy by defining
\[ C_n(s, t) \overset{\text{def}}{=} c(0) + \int_0^s (\cos(G_n(\theta, t)), \sin(G_n(\theta, t))) \, d\theta \]

note that \( C_n(s, 0) = c(s) \) and \( C_n(s, 1) \) is a smooth closed curve.

We now compute the \( \tilde{H}^1 \) action of \( C_n \),
\[ E_{\tilde{H}^1}(C_n) \overset{\text{def}}{=} \int_0^1 \| \partial_t C_n \|^2_{H^1} \, dt = \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 + |D_s \partial_t C_n|^2 \, ds \, dt \]

Since any \( C_n(\cdot, t) \) is by arc parameter, then \( D_s \partial_t C_n = \partial_s C_n \) so
\[ D_s \partial_t C_n = \frac{N(s)}{\|\partial_t C_n\|} \partial_t G_n(s, t) \]

where
\[ N(s) \overset{\text{def}}{=} (- \sin(G_n(s, t)), \cos(G_n(s, t))) \]
is the normal to the curve; so the second term in the action \( E_{\tilde{H}^1}(C_n) \) is exactly equal to \( E_S(G_n) \), that is,
\[ E_{\tilde{H}^1}(C_n) = \int_0^1 \| \partial_t C_n \|^2_{H^1} \, dt = \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 \, ds \, dt + E_S(G_n) \]

We can also prove that \( \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 \, ds \, dt \rightarrow 0 \), so \( E_{\tilde{H}^1}(C_n) \rightarrow 0 \), and then
\[ \lim_{n \to \infty} Len_{\tilde{H}^1}(C_n) = 0 \]

As a second step, to conclude, we assume that \( c \) is flat, that is, the image of \( c \) is contained in a line in the plane; then, up to translation and rotation,
\[ c(\theta) = (c_1(\theta), 0) \]
since \( c \) is by arc parameter, \( c_1 = \pm 1 \). Let then \( f : [0, 2\pi] \) be smooth and with support in \([1, 3]\) and \( f(2) = 1 \); let moreover
\[ C(\theta, t) \overset{\text{def}}{=} (c_1(\theta), tf(\theta)) \]

so
\[ |\partial_\theta C| = \sqrt{1 + (f'(\theta))^2} \geq 1 \]
and then, by direct computation, we can prove that
\[ Len_{\tilde{H}^1}(C) < \infty \]
moreover, any curve \( C(\cdot, t) \) for \( t > 0 \) is not flat, so it can be approximated by smooth curves
§3.ii The completion of \( M \) according to \( H^2 \) distance

Let \( d(c_0, c_1) \) be the geometric distance induced by \( H^2 \) on \( M \) (as defined in 12). Let \( E(c) \equiv \int |D^2_c|^2 \, ds \) be defined on non-constant smooth curves. We prove that

**Theorem 29**  
\( i) \). \( E \) is locally Lipschitz in \( M \) w.r.t. \( d \); and the local Lipschitz constant depends on the length of \( c \).

\( ii) \). As a corollary, all non-constant curves in the completion of \( C^\infty(S^1 \rightarrow \mathbb{R}^n) \) according to the metric \( H^2 \) admit curvature as a measurable function, and the energy of the curvature is finite, that is, \( E(c) < \infty \).

\( iii) \). Vice versa, any non-constant curve admitting curvature in a weak sense and satisfying \( E(c) < \infty \) is approximable by smooth curves.

The rest of this section is devoted to proving the above three statements.

**Proof of 29.i.** Fix a curve \( c_0 \); let \( L_0 \equiv \text{len} c_0 \) be its length.

By eqn. (21) and Proposition 21, we know that the “length function” \( c \mapsto \text{len}(c) \) is Lipschitz in \( M \) w.r.t. the distance \( d \), that is,

\[ |\text{len} c_0 - \text{len} c_1| \leq a_1 d(c_0, c_1) \]

where \( a_1 \) is a positive constant (dependent on \( \lambda \)).

Choose any \( c_1 \) with \( d(c_0, c_1) < L_0/(4a_1) \).

Let \( C(\theta, t) \) be a time varying smooth homotopy connecting \( c_0 \) to (a reparametrization of) \( c_1 \); choose it so that \( \text{Len} C < 2d(c_0, c_1) \); then \( \text{Len} C < L_0/(2a_1) \).

Let \( L(t) \equiv \text{len} C(\cdot, t) \) be the length of the curve at time \( t \). Since at all times \( t \in [0, 1] \), \( d(c_0, C(\cdot, t)) < L_0/(2a_1) \), then \( |L(t) - L_0| < a_1 L_0/(2a_1) = L_0/2 \); in particular,

\[ L_0/2 < L(t) < L_03/2 \]

By using this last inequality, we are allowed to discard \( L(t) \) in most of the following estimates.

We call \( \|f\| \equiv \sqrt{\int |f(s)|^2 \, ds} \) and

\[ N(t) \equiv \|D^2_\theta \partial_t C(\cdot, t)\| = \sqrt{\int |D^2_\theta \partial_t C|^2 \, ds} \]

for convenience; using this notation, we recall that

\[ \|\partial_t C\|_{H^2} = \sqrt{\lambda L(t)^3 N(t)^2 + \frac{1}{L(t)} \|\partial_t C\|^2} ; \]

so \( \|\partial_t C\|_{H^2} \geq \sqrt{\lambda L^{3/2} N(t)} \).

Up to reparameterization in the \( t \) parameter, we can suppose that the path \( t \mapsto C(\cdot, t) \) in \( M \) is by (approximate) arc parameter, that is \( \|\partial_t C\|_{H^2} \) is (almost) constant in \( t \); so we assume, with no loss of generality, that \( \|\partial_t C\|_{H^2} \leq 2d(c_0, c_1) \) for all \( t \in [0, 1] \), and then \( N(t) \leq a_2 d(c_0, c_1) \) where \( a_2 = 2/\sqrt{(L_0/2)^3} \).

We want to prove that

\[ E(c_1) - E(c_0) \leq a_3 d(c_0, c_1) \]
where the constant $a_5$ will depend on $L_0$ and $\lambda$.

By direct computation

$$
\partial_t E(C(\cdot, t)) = \int |D_s^2 C|^2 \langle D_s, \partial_t C, D_s C \rangle \, ds + 2 \int \langle D_s^2 C, \partial_t D_s^2 C \rangle \, ds
$$

we deal with the two addenda in this way:

i). by Poincaré inequality (15) we deduce

$$
sup_\theta |D_s \partial_\theta C| \leq \frac{1}{2} \int |D_s^2 \partial_\theta C| \, ds \leq \sqrt{L(t)} \sqrt{\int |D_s^2 \partial_\theta C|^2 \, ds} = \sqrt{L(t)} N(t)
$$

since $D_s \partial_\theta C = 0$.

So we estimate the first term as

$$
\int |D_s^2 C|^2 \langle D_s, \partial_\theta C, D_s C \rangle \, ds \leq E(C) \sqrt{L(t)} N(t).
$$

ii). The commutator of $D_s$ and $\partial_t$ is $\langle D_s, \partial_\theta, D_s C \rangle D_s$: indeed

$$
\partial_t D_s = \frac{1}{|\partial_\theta C|} \partial_\theta \partial_t + \left(\frac{\partial_t}{|\partial_\theta C|}\right) \partial_\theta = D_s \partial_t - \langle D_s, \partial_\theta, D_s C \rangle D_s
$$

so

$$
\partial_t D_s^2 C = D_s \partial_t D_s C - \langle D_s, \partial_\theta C, D_s C \rangle D_s^2 C =
$$

$$
D_s^2 \partial_t C - D_s(\langle D_s, \partial_\theta C, D_s C \rangle D_s C) - \langle D_s, \partial_\theta C, D_s C \rangle D_s^2 C =
$$

$$
D_s^2 \partial_t C - (D_s^2 \partial_\theta C, D_s C) D_s C - (D_s, \partial_\theta C, D_s^2 C) D_s C -
$$

$$
-2 \langle D_s, \partial_\theta C, D_s C \rangle D_s^2 C
$$

so (since $|D_s C| = 1$)

$$
||\partial_t D_s^2 C|| \leq 2 ||D_s^2 \partial_\theta C|| + 3 ||D_s^2 C|| sup |D_s \partial_\theta C|
$$

that yields an estimate of the second term

$$
\int \langle D_s^2 C, \partial_t D_s^2 C \rangle \, ds \leq \sqrt{E(C)} \left(2N(t) + 3 \sqrt{E(C)} \sqrt{L(t)} N(t) \right)
$$

by using Cauchy-Schwartz.

Summing up

$$
||\partial_t E(C(\cdot, t))|| \leq 2 \sqrt{E(C)} N(t) + 4E(C) \sqrt{L(t)} N(t)
$$

or, since $\sqrt{x} \leq 1 + x$,

$$
||\partial_t E(C(\cdot, t))|| \leq 2N(t) + 2E(C) N(t) + 4E(C) \sqrt{L(t)} N(t)
$$

We recall that $N(t) \leq a_2 d(c_0, c_1)$, $L(t) \leq L_0^3/2$, so we rewrite the above as

$$
||\partial_t E(C(\cdot, t))|| \leq 2a_2 d(c_0, c_1) + 2E(C) a_2 d(c_0, c_1) + 4E(C) a_4 a_2 d(c_0, c_1)
$$
with \( a_4 = \sqrt{L_0 3^{5/2}} \). Apply Gronwall’s Lemma to obtain

\[
E(c_1) \leq \left( E(c_0) + 2a_2 d(c_0, c_1) \right) \exp \left( (2 + 4a_4) a_2 d(c_0, c_1) \right).
\]

Let

\[
g(y) = \left( E(c_0) + 2a_2 y \right) \exp \left( (2 + 4a_4) a_2 y \right)
\]

then \( E(c_1) \leq g(d(c_0, c_1)) \); since \( g \) is convex, and \( g(0) = E(c_0) \), then there exists a \( a_5 > 0 \) such that \( g(y) \leq E(c_0) + a_5 y \) when \( 0 \leq y \leq L_0/(4a_1) \); since we assumed that \( d(c_0, c_1) < L_0/(4a_1) \), then

\[
E(c_1) \leq E(c_0) + a_5 d(c_0, c_1).
\]

Note that \( a_5 \) is ultimately dependent on \( L_0 \) and \( \lambda \).

This ends the proof of the first statement of 29.

**Proof of 29.ii.** To prove the second statement, let \((c_n)_{n \in \mathbb{N}}\) be a Cauchy sequence. Since \( d_{H^1} \leq d_{H^2} \), then as in the proof of 26, we assume that, up to reparameterization and a choice of subsequence, \( c_n \) converges uniformly to a Lipschitz curve \( c \).

Let \( L_0 = \text{len}(c) \). We have assumed in the statement that \( c \) is non-constant; then \( L_0 > 0 \).

Again, the “length function” \( c \mapsto \text{len}(c) \) is Lipschitz, so we know that the sequence \( \text{len}(c_n) \) is Cauchy in \( \mathbb{R} \), so it converges; moreover the “length function” \( c \mapsto \text{len}(c) \) is lower semicontinuous w.r.t. uniform convergence, so \( \lim_n \text{len}(c_n) \geq \text{len}(c) > 0 \). So we assume, up to a subsequence, that \( 2L_0 \geq \text{len}(c_n) \geq L_0 \).

We proved above that, in a neighbourhood of \( c \) of size \( L_0/(8a_1) \), the function \( E(c) = \int |D^2 \theta|^2 \, ds \) is Lipschitz; so we know that the sequence \( E(c_n) \) is bounded, and then (since curves are by arc parameter and \( \text{len}(c_n) \geq L_0 \)) the energy \( \int |D^2 \theta|^2 \, ds \) is bounded: then \( \partial_\theta c_n \) are uniformly Hölder continuous, so by Ascoli-Arzelà compactness theorems, up to a subsequence, \( \partial_\theta c_n(\theta) \) converges.

As a corollary we obtain that \( \lim_n \text{len}(c_n) = \text{len}(c) \), that \( c \) is parameterized by arc parameter, and that \( D_n c_n(\theta) \) converges to \( D c(\theta) \).

Since the functional \( \int |D^2 \theta|^2 \, ds \) is bounded in \( n \), then by a theorem in [1], \( c \) admits weak derivative \( \partial_\theta^\ast c \) and \( \int |\partial_\theta^\ast c|^2 \, ds \leq \infty \), and equivalently, \( \int |D^2 \theta|^2 \, ds \leq \infty \).

**Proof of 29.iii.** For the third statement, viceversa, let \( c \) be a rectifiable curve, and assume that it is non-constant, and \( E(c) < \infty \). Since the metric is independent of rescaling, we rescale \( c \), and assume that \( |\partial_\theta c(u)| = 1 \).

We express in Fourier series

\[
c(u) = \sum_{n \in \mathbb{Z}} l_n \exp(inu) \tag{24}
\]

(by equating \( S^1 = \mathbb{R}/2\pi \)), then we decide that

\[
C(u, t) = \sum_{n \in \mathbb{Z}} l_n \exp(inu - f(n)t) \tag{25}
\]

with \( f(n) = f(-n) \geq 0 \) and \( \lim f(n)/\log(n) = \infty \); (for example, \( f(n) = |n| \) or \( f(n) = (\log(|n| + 2))^2 \)): then \( C(\cdot, t) \) is smooth for any \( t > 0 \).

We want to prove that, for \( t \) small, \( C(\cdot, t) \) is near \( c \) in the \( H^2 \) metric; to this end, let \( \tilde{C} \) be the linear interpolator

\[
\tilde{C}(u, t, \tau) = \left( 1 - \tau \right) c(u) + \tau C(u, t) = \sum_{n \in \mathbb{Z}} l_n e^{inu} \left( 1 - \tau + \tau e^{-f(n)t} \right) \tag{26}
\]
we will prove that
\[
\int_0^1 \left( \int_{S^1} |\partial_u \tilde{C}|^2 \, ds + \lambda L^4 \int_{S^1} |D_s^2 \partial_\tau \tilde{C}|^2 \, ds \right) \, d\tau < \delta(t) \tag{27}
\]
where \( \lim_{t \to 0} \delta(t) = 0 \), and \( L \) is the length of \( \tilde{C}(\cdot, t, \tau) \).

We need some preliminary results:

• we prove that
\[
\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 \, du < \delta_1(t) \tag{28}
\]
where \( \lim_{t \to 0} \delta_1(t) = 0 \), uniformly in \( \tau \in [0, 1] \); we write
\[
\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 \, du = 2\pi \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 (1 - e^{-f(n)t})^2
\]
and since
\[
2\pi \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 = E(c) = \int_{S^1} |\partial_{uu} c|^2 \, ds < \infty
\]
and \( \lim_{t \to 0} (1 - e^{-f(n)t})^2 = 0 \), we can apply Lebesgue dominated convergence theorem.

• We prove that
\[
|\partial_u c - \partial_u \tilde{C}| < \delta_2(t) \tag{29}
\]
where \( \lim_{t \to 0} \delta_2(t) = 0 \), uniformly in \( \tau \in [0, 1] \); indeed
\[
|\partial_u c - \partial_u \tilde{C}| \leq \tau \sum_{n \in \mathbb{Z}} |l_n| |n|(1 - e^{-f(n)t}) \leq \sqrt{\sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4} \sqrt{\sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n^2} (1 - e^{-f(n)t})^2}
\]
again we apply Lebesgue dominated convergence theorem.

• By the above we also obtain that for \( t \) small,
\[
3/2 \geq |\partial_u \tilde{C}| \geq 1/2 \text{ uniformly in } \tau, u \tag{30}
\]

• We can similarly prove that
\[
|c - \tilde{C}| < \delta_3(t) \tag{31}
\]

By direct computation
\[
D_s^2 \partial_\tau \tilde{C} = \frac{\partial_{uu\tau} \tilde{C}}{|\partial_u \tilde{C}|^2} + \frac{\langle \partial_{uu} \tilde{C}, \partial_u \tilde{C} \rangle \partial_{u\tau} \tilde{C}}{|\partial_u \tilde{C}|^4}
\]
but then, for \( t \) small, by (30),
\[
|D_s^2 \partial_\tau \tilde{C}| \leq 4 |\partial_{uu\tau} \tilde{C}| + 24 |\partial_{uu} \tilde{C}| |\partial_{u\tau} \tilde{C}|
\]
We use the fact that
\[ \partial_{uu} \tilde{C} = \partial_{uu} C - \partial_{u} c , \quad \partial_{u \tau} \tilde{C} = \partial_{u} C - \partial_{u} c , \quad \partial_{\tau} \tilde{C} = C - c , \]
so by eqn. (28) and eqn. (29)
\[ \int \int |D^2_{\tau} \partial_{\tau} \tilde{C}|^2 \, ds \, d\tau \leq a_1(\delta_1(t)) + E(c) \delta_2(t) \]
and by eqn. (31) \( \int \int |\partial_{\tau} \tilde{C}|^2 \, ds \, d\tau \leq 8 \delta_3(t) \). Eventually we combine all above to bound eqn. (27) by setting \( \delta(t) = a_2 \delta_3(t) + \lambda a_2(\delta_1(t) + E(c) \delta_2(t)) \).

This concludes the proof.

References


REFERENCES


